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On local matchability in groups and vector spaces *†

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Abstract

In this paper, we define locally matchable subsets of a group which is extracted from the concept of matchings in groups and used as a tool to give alternative proofs for existing results in matching theory. We also give the linear analogue of local matchability for subspaces in a field extension. Our tools mix additive number theory, combinatorics and algebra.

1 Introduction

The notion of matchings in groups was used to study an old problem of Wakeford concerning canonical forms for symmetric tensors [11]. Losonczy in [10]

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introduced matchings for groups to work on Wakeford's problem. Let G be an additive abelian group and A and B be two non-empty subsets of G. A *matching* from A to B is a map $f : A \to B$ which is bijective and satisfies the condition $a + f(a) \notin A$, for any $a \in A$. This concept can be used to define the matching property in the sense of groups. It is said that the abelian group G has the *matching property*, if for every pair A and B of non-empty finite subsets satisfying $\#A = \#B$ and $0 \notin B$, there exists at least one matching from A to B. In addition to this generalization of the definition of matching, we can see the following which is its extension in a new fashion:

Let A, B and C be non-empty finite subsets of G such that $A \subseteq C$. A C-*matching* from A to B is a map $f : A \to B$ which is bijective and satisfies the condition $a + f(a) \notin C$, for all $a \in A$. Clearly, a matching from A to B is an A-matching from A to B.

The following definition can also be extracted from the original concept of matchable subsets of a group.

Let A and B be non-empty subsets of G such that for any proper subgroup H of G with $H \cap B \neq \emptyset$ and $a + H \subseteq A$ for some $a \in A$, one can find an A-matching from a subset of A to $H \cap B$. Then we say A is locally matched to B.

It is obvious that if A is matched to B, then it is locally matched to B as well. We will see in the section 3 that matchability concludes local matchability, i.e. these two concepts are equivalent. We will use this concept as a tool to give an alternative proof for groups with matching property.

Organization of the paper

The purpose of this paper is to find the relations between local matchability and matchability in groups and vector spaces to give alternative proofs for existing results on matching property for groups and also its linear analogue. In section 2, we will mention the results that have been proven on matching in groups and vector spaces and also some tools of additive number theory required to prove our main results. In section 3, we will show the equivalency between matchability and local matchability for subsets of a group. Section 4 concerned with the linear analogue of one of Losonzy's results on matchings for cyclic groups. Furthermore, we will present a dimension criteria for the primitive subspaces related to our results in Theorem 4.1. Finally, in section 5, we show that if A is matched to B, then A it is locally matched to B in the sense of vector spaces. However, the converse is still not clear in the general case. We will see in Theorem 5.2 that for vector spaces in a field extension whose algebraic elements are separable, the local matchability implies the matchability. As an application we give an alternative proof for author's main result in [6].

2 Prelimanaries

The following theorems are known about matching for groups and vector spaces. As we already mentioned, our goal in this paper is to prove their more general cases.

Theorem 1. *An abelian group G has the matching property if and only if G is torsion-free or cyclic of prime order [8].*

Theorem 2. *Let G be a non-trivial finite cyclic group. Suppose we are given non-empty subsets A and B of G such that $\#A = \#B$ and every element of B is a generator of G. Then there exists at least one matching from A to B [8].*

Here, we define the matching property for subspaces in a field extension. Let $K \subseteq L$ be a field extension and A and B be n-dimensional K-subspaces of the field extension L. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be bases of A and B, respectively. It is said that $\mathcal{A}$ is *matched* to $\mathcal{B}$ if

$$a_i b \in A \;\Rightarrow\; b \in \langle b_1, \ldots, \hat{b}_i, \ldots, b_n \rangle,$$

for all $b \in B$ and $i = 1, \ldots, n$, where $\langle b_1, \ldots, \hat{b}_i, \ldots, b_n \rangle$ is the hyperplane of B spanned by the set $\mathcal{B} \setminus \{b_i\}$; moreover, it is said that A is *matched* to B if every basis $\mathcal{A}$ of A can be matched to a basis $\mathcal{B}$ of B. As it is seen, the matchable bases are defined in a natural way based on the definition of matching in a group. Indeed, we can consider $\mathcal{A}$ and $\mathcal{B}$ as subsets of the multiplicative group L^* and so the bijection $a_i \mapsto b_i$ is a matching in the group setting sense. It's said L has the *linear matching property* if, for every $n \geq 1$ and every n-dimensional subspaces A and B of L with $1 \notin B$, the subspaces A is matched with B.

Now, our definition for matchable subsets of two matchable bases:

Let $\tilde{A}$ and $\tilde{B}$ be two K-subspaces of A and B, respectively. We say that $\tilde{A}$ is A-matched to $\tilde{B}$, if for any basis $\tilde{\mathcal{A}} = \{a_1, \ldots, a_m\}$ of $\tilde{A}$, there exists a basis $\tilde{\mathcal{B}} = \{b_1, \ldots, b_m\}$ of $\tilde{B}$ for which $a_i b_i \notin A$, for $i = 1, \ldots, m$. In this case, it is also said that $\tilde{\mathcal{A}}$ is A-matched to $\tilde{\mathcal{B}}$.

The following is the linear analogue of locally matchable subsets for the vector spaces in a field extension.

Let $K \subseteq L$ be a field extension and A, B be two n-dimensional K-subspaces of L. We say that A is locally matched to B if for any proper subfield H of L

with $H \cap B \neq \{0\}$ and $aH \subseteq A$, for some $a \in A$, one can find a subspace $\tilde{A}$ of A such that $\tilde{A}$ is A-matched to $H \cap B$.
The following theorem is a dimension criteria for matchable bases. For more results on linear version of matching see [2, 3, 4 and 6].

Theorem 3. *Let $K \subset L$ be a field extension and A and B be two n-dimensional K-subspaces of L. Suppose that $\mathcal{A} = \{a_1, \ldots, a_n\}$ is a basis of A. Then $\mathcal{A}$ can be matched to a basis of B if and only if, for all $J \subseteq \{1, \ldots, n\}$, we have:*

$$\dim \bigcap_{i \in J} \left(a_i^{-1} A \cap B \right) \leq n - \#J.$$

See [6] for more details. One of the main result in [6] is that a field extension $K \subset L$ has the linear matching property if and only if there are no trivial finite intermediate extension $K \subset M \subset L$. We would like to mention that, although the statement of [6, Theorem 2.6] is slightly different and assumes that the extension is either purely transcendental or finite of prime degree, what they actually use in their proof is that there are no nontrivial finite intermediate extensions, which is a weaker condition. (See also [1].)

For proving our main results, we shall need the following theorems from [9, page 116, Theorem 4.3].

Theorem 4 (Kneser)**.** *If $C = A + B$, where A and B are finite subsets of an abelian group G, then*

$$\#C \geq \#A + \#B - \#H,$$

where H is the subgroup $H = \{g \in G : \ C + g = C\}$.
See [5] for more details regarding the following theorem which is the linear analogue of Kneser's theorem.

Theorem 5. *Let $K \subseteq L$ be a field extension in which every algebraic element of L is separable over K. Let $A, B \subset L$ be non-zero finite-dimensional K-subspaces of L and H be the stabilizer of $\langle AB \rangle$. Then*

$$\dim_K \langle AB \rangle \geq \dim_K A + \dim_K B - \dim_K H.$$

(Here H is a subfield of L containing K and we have $H\langle AB \rangle = \langle AB \rangle$.)

Let E be a vector space over the field K and let $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ be a collection of vector subspaces of E. A free transversal for $\mathcal{E}$ is a free family of vectors $\{x_1, \ldots, x_n\}$ in E satisfying $x_i \in E_i$ for all $i = 1, \ldots, n$. The following result of Rado [8] gives necessary and sufficient conditions for the existence of a free transversal for $\mathcal{E}$, very similar to those of Hall's marriage theorem. The interested readers are also referred to [7].

Theorem 6. *Let E be a vector space over K and let $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ be a family of vector subspaces of E. Then $\mathcal{E}$ admits a free transversal if and only if*

$$\dim_K +_{i \in J} E_i \geq \# J,$$

for all $J \subseteq \{1, \ldots, n\}$.

We shall use the following standard notation denoted by

$$B^* = \{f : B \to K : \ f \text{ is linear}\},$$

the dual of B where B is a K-vector space; furthermore, for any subspace $C \subseteq B$, we denote by

$$C^{\perp} = \{f \in B^* : \ C \subseteq \ker f\}$$

the orthogonal of C in B^*. Recall that $\dim_K C^{\perp} = \dim_K B - \dim_K C$.

Corollary 1. *Let E be a n-dimensional vector space over K and $\mathcal{E} = \{E_1, \dots, E_k\}$ be a family of vector subspaces of E such that for any $J \subseteq \{1, \dots, k\}$, $\dim_K \bigcap_{i \in J} E_i \leq n - \#J$. Then, there exist vector subspaces $\tilde{E}_i$ of E such that $E_i \subseteq \tilde{E}_i$ for $i = 1, \dots, k$ and for any $J \subseteq \{1, \dots, k\}$, $\dim_K \bigcap_{i \in J} \tilde{E}_i = n - \#J$.*

Proof. Since $\dim_K \bigcap_{i \in J} E_i \leq n - \#J$, then $\dim_K +_{i \in J} E_i^{\perp} \geq J$. Using Theorem 2.6, $\tilde{\mathcal{E}} = \left\{E_1^{\perp}, \dots, E_k^{\perp}\right\}$ admits a free transversal. Let $(a_1, \dots, a_k)$ be a free transversal for $\tilde{\mathcal{E}}$. Then $\dim_K +_{i \in J} \langle a_i \rangle = \#J$. Set $\tilde{E}_i = \langle a_i \rangle^{\perp}$, thus $E_i \subseteq \tilde{E}_i$ and $\dim_K \bigcap_{i \in J} \tilde{E}_i = n - \#J$. $\square$

We finish this section by inclusion-exclusion principle for vector spaces which states that for two finite dimensional K-vector spaces A and B, we have the following equality for the dimension of K-vector space $A + B$:

$$\dim_K(A + B) = \dim_K A + \dim_K B - \dim_K A \cap B.$$

Note that since $\dim_K(A + B) = \dim_K \langle A \cup B \rangle$, then we can rewrite the above equality as follows:

$$\dim_K \langle A \cup B \rangle = \dim_K A + \dim_K B - \dim_K(A \cap B).$$

3 Local matchability for groups

The following theorem shows that local matchability is equivalent to matchability for abelian groups. Note that it is obvious that matchability implies local matchability. Then, we just show its converse.

Theorem 7. *Let G be an additive abelian group and A, B be non-empty finite subsets of G satisfying the conditions $\#A = \#B$ and $0 \notin B$. If A is locally matched to B, then A is matched to B.*

Proof. Suppose there is no matching from A to B. We are going to reach a contradiction. By Hall marriage Theorem there exists a non-empty finite subset S of A such that $\#B \setminus U < \#S$, where $U = \{b \in B : \; s + b \in A$, for any $s \in S\}$. Let $\#A = \#B = n$, then $\#U + \#S > n$. Set $U_0 = U \cup \{0\}$. Using Kneser's Theorem one can find the subgroup H of G such that

$$\#(U_0 + S) \geq \#U_0 + \#S - \#H, \tag{1}$$

where $H = \{g \in G : \; g + U_0 + S = U_0 + S\}$. Applying Kneser's Theorem for $U' = H \cup U$ and S, we can find the subgroup H' of G for which

$$\#(U' + S) \geq \#U' + \#S - \#H', \tag{2}$$

where $H' = \{g \in G : \; g + U' + S = U' + S\}$. We claim that $H = H'$ and to prove this, it suffices to show that $U' + S = U_0 + S$. We have

$$\begin{aligned} U' + S &= (H \cup U) + S = (H + S) \cup (U_0 + S) \\ &= (H + S) \cup (U_0 + S + H) \\ &= H + (S \cup (U_0 + S)) \\ &= H + (U_0 + S) = U_0 + S. \end{aligned} \tag{3}$$

Then $H = H'$ and it follows from (2) that

$$\#(U_0 + S) \geq \#U' + \#S - \#H. \tag{4}$$

Using (3), (4) we get

$$\begin{aligned} \#(U_0 + S) &= \#(U' + S) \\ &= \#U' + \#S - \#H \\ &= \#(H \cup U) + \#S - \#H \\ &= \#H + \#U - \#(H \cap U) + \#S - \#H \\ &= \#U + \#S - \#(H \cap U). \end{aligned} \tag{5}$$

As $U_0 + S = S \cup (S + U)$, (5) follows

$$\#(S \cup (S + U)) \geq \#U + \#S - \#(H \cap U). \tag{6}$$

Now, we have two cases for $H \cap U$.

1. If $H \cap U$ is empty, then by (6) we conclude that $\#(S \cup (S + U)) \geq n$. On the other hand $S \cup (S + U)$ is a subset of A. We would have $\#A > n$, which contradicts $\#A = n$ above.

2. If $H \cap U$ is non-empty, then $H \cap B$ is. Also, if $s \in S \subseteq A$, then according to the definition of H, $s + H \subseteq U_0 + S + H = U_0 + S \subseteq A$. As A is locally matched to B, then there is an A-matching f from a subset of A to $H \cap B$. We claim that $f^{-1}(H \cap U) \cap (U_0 + S)$ is empty. If not and $a \in f^{-1}(H \cap U) \cap (U_0 + S)$, then $a + f(a) \in (U_0 + S) + H$ as $a \in U_0 + S$ and $f(a) \in H \cap U \subseteq H$. Since $U_0 + S \subseteq A$, then $a + f(a) \in A$ which contradicts the case f being an A-matching. Therefore $f^{-1}(H \cap U) \cap (U_0 + S)$ is empty. As the sets $f^{-1}(H \cap U)$ and $U_0 + S$ are both subsets of A and have nothing in common, then $\#f^{-1}(H \cap U) + \#(U_0 + S) \leq n$. Thus $\#(H \cap U) + \#(U_0 + S) \leq n$ and this tells us $\#(H \cap U) + \#(S \cup (S + U)) \leq n$. Next, using (6) yields that $\#U + \#S \leq n$ which is a contradiction.

Therefore in both cases we extract contradictions. Then there is a matching from A to B. $\square$

Using Theorem 3.1, we give an alternative proof to Theorem 2.1.

Assume that G is either torsion-free or cyclic of prime order. Then G has no non-trivial subgroup of finite order. This means if $A, B \subseteq G$ with $|A| = |B|$ and $0 \notin B$, then A is locally matched to B. Using Theorem 3.1 yields that A is matched to B and so G has matching property.

Conversely, assume that G is neither torsion-free nor cyclic of prime order. Then it has a non-trivial finite subgroup H. Choose $g \in G \setminus H$ and set $A = H$ and $B = H \cup \{g\} \setminus \{0\}$. Clearly, $H \cap B \neq \emptyset$ and $a + H \subseteq A$ for some $a \in A$ (Indeed for any $a \in A$). If A is locally matched to B, then one can find an A-matching f from a subset A_0 of A to $H \cap B$. But if $a \in A_0$, then $a + f(a) \in H + (H \cap B) = H + (H \setminus \{0\}) = H = A$, which is a contradiction. Then A is not locally matched to B and so by Theorem 3.1, A is not matched to B. Therefore G has no matching property.

4 The linear analogue of Losonzcy's result on matchable subsets

The following is the linear version of Theorem 2.2 which investigates the matchable subspaces in a simple field extension. Here, we say that $K \subseteq L$ is a simple field extension if $L = K(\alpha)$, for some $\alpha \in L$. Also, if B is a K-subspace of L suth that $K(b) = L$, for any $b \in B \setminus \{0\}$, we say that B is a primitive K-subspace of L.

Theorem 8. *Let $K \subsetneqq L$ be a finite and separable field extension and A and B be two n-dimensional K-subspaces in L with $n \geq 1$ and B is a primitive K-subspace of L. Then A is matched with B.*

Proof. Assume that A is not matched to B. Using Theorem 2.3, one can find $J \subseteq \{1, \ldots, n\}$ and a basis $\mathcal{A} = \{a_1, \ldots, a_n\}$ of A such that

$$\bigcap_{i \in J} \left(a_i^{-1} A \cap B\right) > n - \#J. \tag{7}$$

Set $S = \langle a_i : \ i \in J \rangle$ the K-subspace of A spanned by a_i's, $i \in J$, $U =$

$\bigcap_{i\in J}\left(a_i^{-1}A\cap B\right)$ and $U_0 = U\cup\{1\}$. Now, by Theorem 2.5 one can find a subfield H of L such that

$$\dim_K\langle U_0S\rangle \;\geq\; \dim_K U_0 + \dim_K S - \dim_K H,$$

where H is the stabilizer of $\langle U_0S\rangle$, i.e. $H = \{x\in L:\ x\langle U_0S\rangle \subseteq \langle U_0S\rangle\}$. Define $U' = \langle H\cup U\rangle$. Using Theorem 2.5 again, we have

$$\dim_K\langle U'S\rangle \geq \dim_K\langle U'\rangle + \dim_K\langle S\rangle - \dim_K H',$$

where H' is the stabilizer of $\langle U'S\rangle$. Next, we have

$$\begin{aligned}\langle U'S\rangle &= \langle (H\cup U)S\rangle = \langle HS\cup U_0S\rangle\\ &= \langle HS\cup HU_0S\rangle = H\langle S\cup U_0S\rangle\\ &= H\langle U_0S\rangle = \langle U_0S\rangle.\end{aligned} \tag{8}$$

This follows $H = H'$ and then

$$\dim_K\langle U'S\rangle \geq \dim_K\langle U'\rangle + \dim_K S - \dim_K H. \tag{9}$$

Using (8) and (9), we have

$$\begin{aligned}\dim_K\langle U_0S\rangle &\geq \dim_K U' + \dim_K S - \dim_K H\\ &= \dim_K\langle H\cup U\rangle + \dim_K S - \dim_K H.\end{aligned} \tag{10}$$

Using (10), the fact that $\langle U_0S\rangle = \langle S\cup SU\rangle$ and inclusion-exclusion principle for vector spaces we have:

$$\begin{aligned}\dim_K\langle S\cup SU\rangle &\geq \dim_K\langle H\cup U\rangle + \dim_K S - \dim_K H\\ &= \dim_K H + \dim_K U - \dim_K(H\cap U) + \dim_K S - \dim_K H\\ &= \dim_K U + \dim_K S - \dim_K(H\cap U).\end{aligned} \tag{11}$$

Now, we have two cases for the subspace $H\cap U$.

1. If $H \cap U = \{0\}$, then by (7) and (11) we get $\dim_K \langle S \cup SU \rangle \geq n$ and this is impossible as $S \cup SU \subseteq A$ and $\dim_K A = n$.

2. If $H \cap U \neq \{0\}$, then $H \cap B \neq \{0\}$ and since B is a primitive subspace of L, then $H = L$. By the definition of U and S, $HUS \subseteq A$ and this follows $LUS \subseteq A$ and so $A = L$. Then $B = L$ as $\dim_K A = \dim_K B$ and this means $K \subseteq B$. Therefore if $a \in K \setminus \{0\}$, then $K = K(a) = L$, which is impossible.

In both cases, we extract contradictions. Then A is matched to B. $\square$

We discussed the matching property for the subspaces of a simple field extension whose non-zero elements generates the whole field in Theorem 4.1. We call such a vector space as a primitive subspace of the field extension. Now, this question can be asked that how large can this vector space be? We answer to this question in the following theorem:

Theorem 9. *Let K be an infinite field and $K \subsetneq L$ be a finite simple field extension. Then we have:*

$$n = n(K, L) + m(K, L),$$

where $n = [L : K]$, $n(K, L) = \max\{[F : K];\ K \subseteq F \subsetneq L \text{ is a filed}\}$ and $m(K, L) = \max\{\dim_K V;\ K(a) = L, \text{ for any } a \in V \setminus \{0\}\}$.

Before proving the above theorem, we mention a well-known theorem of elementary linear algebra that if V is a finite dimensional vector space over an infinite field K, then V can not be written as a finite union of its proper subspaces. Also, we use Atin's theorem on field extensions which states that if $K \subseteq L$ is a finite field extension, then it is simple, i.e. $L = K(\alpha)$, for some

$\alpha \in L$ if and only if there exist only finitely many intermediate subfields F of $K \subseteq L$.

Proof. As $K \subseteq L$ is a simple field extension, by Artin's theorem on primitive elements it has only finitely many intermediate subfields. Let $\{F_i\}_{i=1}^{m}$ be the family of the all proper intermediate subfields of $K \subseteq L$. By the above-mentioned theorem, we get $\bigcup_{i=1}^{m} F_i \neq L$.

Now, without loss if generality, we may assume that $n(K, L) = [F_1 : K]$. Choose $a_1 \in L \setminus \bigcup_{i=1}^{m} F_i$ and define $F_i^{(1)} = F_i \oplus \langle a_1 \rangle$, for $1 \leq i \leq m$. We break the proof to two cases:

1. If $L = \bigcup_{i=1}^{m} F_i^{(1)}$, then $L = F_j^{(1)}$, for some $1 \leq j \leq m$. Consider the K-subspace $T_1 = \langle a_1 \rangle$ of L, spanned by a_1. We claim that $m(K, L) = 1 = \dim_K T_1$. If $m(K, L) > 1$, so there exists a K-subspace T_1' of L for which $\dim T_1 < \dim T_1'$ and $K(a) = L$, for any $a \in T_1' \setminus \{0\}$. We have $n = \dim_K L = \dim_K(F_j \oplus T_1) \leq \dim_K(F_1 \oplus T_1')$. On the other hand, as $F_1 \oplus T_1' \subseteq L$, then $\dim_K(F_1 \oplus T_1') \leq n$. Consequently, we get $\dim_K(F_1 \oplus T_1') = \dim_K(F_1 \oplus T_1)$. This yields $\dim_K T_1 = \dim_K T_1'$, a contradiction. Then $m(K, L) = 1$ and so $n = [F_1 : K] + \dim_K T_1 = n(K, L) + m(K, L)$.

2. If $L \neq \bigcup_{i=1}^{m} F_i^{(1)}$, choose $a_2 \in L \setminus \bigcup_{i=1}^{m} F_i^{(1)}$ and define $T_2 = \langle a_1, a_2 \rangle$ as a K-subspace of L spanned by a_1 and a_2, and $F_i^{(2)} = F_i \oplus T_2$, $1 \leq i \leq m$. We break this case to two subcases.
   (a) If $L = \bigcup_{i=1}^{m} F_i^{(2)}$, by using the similar argument to the previous case, we get $n = n(K, L) + m(K, L)$.

(b) If $L \neq \bigcup_{i=1}^{m} F_i^{(2)}$, choose $a_3 \in L \setminus \bigcup_{i=1}^{m} F_i^{(2)}$ and define the K-subspace of L $T_3 = \langle a_1, a_2, a_3 \rangle$ generated by a_1, a_2, a_3, and $F_i^{(3)} = F_i \oplus T_3$.

Continuing this procedure, we get two increasing families $\{F_i^{(j)}\}$, $\{T_j\}$ of K-subspaces of L ($1 \leq i \leq m$ and $j \in \mathbb{N}$). Since $[L : K] < \infty$, so $F_i^{\ell} = F_i^{\ell+1} = \cdots$ and $T_\ell = T_{\ell+1} = \cdots$, for some $\ell \in \mathbb{N}$. We get the desired equality. $\square$

We discussed Theorem 4.2 for the finite extensions of infinite fields. The interested reader is encouraged to investigate this equality in the case that the base field is finite.

5 Local matchability for subspaces in a field extension

In this section, Theorem 5.2 and 5.3 give the relation between matchability and local matchability for vector spaces in a field extension. First, we state the following lemma which gives a dimension criteria for vector subspaces of matchable spaces in a field extension.

Lemma 1. *Let $K \subsetneq L$ be a field extension and $A, B \subseteq L$ be two non-zero n-dimensional K-subspaces of L. Assume that H is an intermediate field of $K \subsetneq L$ satisfying the followings conditions:*

1. *$H \cap B \neq \{0\}$;*
2. *$aH \subseteq A$, for some $a \in A$.*

Let $\tilde{\mathcal{B}} = \{b_1, \ldots, b_m\}$ be a basis for $H \cap B$ and $A_b = \{a \in A : \ ab \in A\}$, for any $b \in B$. Then we have

$$\dim_K \bigcap_{i\in J} A_{b_i} \leq n - \#J, \tag{12}$$

for any $J \subseteq \{1, \ldots, m\}$.

Proof. If $\bigcap_{i\in J} A_{b_i} = \{0\}$, then (12) holds. Now, assume that $\bigcap_{i\in J} A_{b_i}$ is non-zero and $\{a_1, \ldots, a_t\}$ is a basis for it. As $a_k \in \bigcap_{i\in J} A_{b_i}$, for all $1 \leq k \leq t$, then $a_k b_i \in A$, for any $i \in J$. This means $b_i \in a_k^{-1}A \cap B$, for all $i \in J$ and so $b_i \in \bigcap_{k=1}^{t} \left(a_k^{-1}A \cap B\right)$. Therefore $\#J \leq \dim_K \bigcap_{k=1}^{t} \left(a_k^{-1}A \cap B\right)$. As A is matched to B, then by Theorem 2.3 we have $\dim_K \bigcap_{k=1}^{t} \left(a_k^{-1}A \cap B\right) \leq n - t$. Therefore, totally we obtain $t \leq n - \#J$ and this means $\dim_K \bigcap_{i\in J} A_{b_i} \leq n - \#J$. $\square$

Now, we are ready to prove our theorem regarding the relation between matchability and local matchability.

Theorem 10. *Let $K, H, L, A, B, \tilde{\mathcal{B}}$ and A_b be as Lemma 5.1. If A is matched to B, then A is locally matched to B.*

Proof. We must prove that there exists a K-subspace $\tilde{A}$ of A such that $\tilde{A}$ is A-matched to $H \cap B$.
Consider the family $\mathcal{E} = \{A_{b_1}, \ldots, A_{b_m}\}$ of vector subspaces of A. Using Corollary 2.7 and Lemma 5.1, there exist subspaces $\tilde{A}_i$ of A for which $A_{b_i} \subseteq \tilde{A}_i$ and $\dim_K \bigcap_{i\in J} \tilde{A}_{b_i} = n - \#J$. Let $\tilde{A}$ be an algebraic complement of $\bigcap_{i=1}^{m} \tilde{A}_i$ in A, i.e. $\tilde{A} \oplus \bigcap_{i=1}^{m} \tilde{A}_i = A$. We prove that $\tilde{A}$ is the desired subspace of A. Namely, we show that it is A-matched to $H \cap B$.
Let $\tilde{\mathcal{A}} = \{a_1, \ldots, a_m\}$ be a basis for $H \cap B$. We claim that there exists

$\sigma \in \mathcal{S}_m$ such that $\tilde{\mathcal{A}}$ is A-matched to $\tilde{\mathcal{B}}_\sigma = \{b_{\sigma(1)}, \ldots, b_{\sigma(m)}\}$. Obvieusly $\bigcap_{i=1}^{m} \tilde{A}_i \cup \left(\bigcap_{i\in J} \tilde{A}_i \cap \tilde{A} \right) \subseteq \bigcap_{i\in J} \tilde{A}_i$ and $\bigcap_{i=1}^{m} \tilde{A}_i \cap \left(\bigcap_{i\in J} \tilde{A}_i \cap \tilde{A} \right) = \{0\}$, for any $J \subseteq \{1, \ldots, m\}$. Then $\dim_K \left(\bigcap_{i\in J} \tilde{A}_i \cap \tilde{A} \right) + \dim_K \bigcap_{i=1}^{m} \tilde{A}_i \leq \dim_K \bigcap_{i\in J} \tilde{A}_i$. So, we actually get:

$$\dim_K \left(\bigcap_{i\in J} \tilde{A}_i \cap \tilde{A} \right) \leq (n - \#J) - (n - m) = m - \#J, \tag{13}$$

for any $J \subseteq \{1, \ldots, m\}$. It follows from (13) that at least $\#J$ elements of $\tilde{\mathcal{A}}$ are not in $\bigcap_{i\in J} \tilde{A}_i \cap \tilde{A}$. Without loss of generality, we assume that $a_1, \ldots, a_{\#J}$ are such elements. Then for any $k \in \{1, \ldots, \#J\}$, one can find $i \in J$ such that $a_k \notin \tilde{A}_i$. Therefore, $a_k \in \tilde{A}_{b_i}$ and so $a_k b_i \notin A$. Using Hall marriage Theorem, there exists a bijection $f : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ such that $a_i f(a_i) \notin A$, for any $1 \leq i \leq m$. Define the permutation $\sigma \in \mathcal{S}_m$ by $\sigma(i) = j$ if $f(a_i) = b_j$, $1 \leq i, j \leq m$. Thus $\tilde{\mathcal{A}}$ is A-matched to $\tilde{\mathcal{B}}_\sigma$, as claimed. □

The following theorem shows that local matchability impleis matchability for subspaces of a field extension whose algebraic elements are separable.

Theorem 11. *Let $K \subseteq L$ be a field extension in which every algebraic element of L is separable over K. Let $A, B \subseteq L$ be two non-zero n-dimensional K-subspaces with $1 \notin B$. If A is locally matched to B, then A is matched to B.*

Proof. Assume to the contrary A is not matched to B. Then, by Theorem 2.3 there exist a basis $\mathcal{A} = \{a_1, \ldots, a_n\}$ of A and $J \subseteq \{1, \ldots, n\}$ such that $\dim \bigcap_{i\in J} (a_i^{-1} A \cap B) > n - \#J$. Set $S = \langle a_i : \ i \in J \rangle$ as a K-subspace of A, $U = \bigcap_{i\in J} (a_i^{-1} A \cap B)$ and $U_0 = \langle U \cup \{1\} \rangle$. Using Theorem 2.5 there exists an

intermediate subfield H of $K \subset L$ such that

$$\dim_K \langle U_0 S \rangle \geq \dim_K U_0 + \dim_K S - \dim_K H \tag{14}$$

, where H is the stabilizer of $\langle U_0 S \rangle$. Define $U' = H \cup U$. Reusing Theorem 2.5, one can find an intermediate subfield H' of $K \subseteq L$ for which

$$\dim_K \langle U' S \rangle \geq \dim_K \langle U' \rangle + \dim_K S - \dim_K H' \tag{15}$$

, where H' is the stabilizer of $\langle U' S \rangle$. The following computations show that $\langle U' S \rangle = \langle U_0 S \rangle$;

$$\begin{aligned} \langle U' S \rangle &= \langle (H \cup U) S \rangle = \langle HS \rangle \cup \langle U_0 S \rangle \\ &= \langle HS \rangle \cup \langle U_0 SH \rangle = H \langle S \cup U_0 S \rangle \\ &= H \langle U_0 S \rangle = \langle U_0 S \rangle. \end{aligned} \tag{16}$$

Then, the stabilizers of these two subspaces must be the same, i.e. $H = H'$. Then we would have

$$\dim_K \langle U' S \rangle \geq \dim_K \langle U' \rangle + \dim_K S - \dim_K H. \tag{17}$$

Bearing (15) and (16) in mind and using inclusion-exclusion principle for vector spaces we get:

$$\begin{aligned} \dim_K \langle U_0 S \rangle &= \dim_K \langle U' S \rangle \\ &\geq \dim_K \langle U' \rangle + \dim_K S - \dim_K H \\ &= \dim_K \langle H \cup U \rangle + \dim_K S - \dim_K H \\ &= \dim_K H + \dim_K U - \dim_K (H \cap U) + \dim_K S - \dim_K H \\ &= \dim_K U + \dim_K S - \dim_K (H \cap U). \end{aligned} \tag{18}$$

Now, we have two cases for $H \cap U$.

1. If $H \cap U = \{0\}$, then $\dim_K \langle S \cup SU \rangle > n$. On the other hand since $S \cup SU \subseteq A$, we would have $\dim_K A > n$, contradicting our assumption $\dim_K A = n$.

2. If $H \cap U$ is a non-zero vector space, then $H \cap B$ so is. It is clear that $aH \subseteq A$, for some $a \in A$ (Indeed, $USH \subseteq A$). Since A is locally matched to B, one can find a subspaces $\tilde{A}$ of A such that $\tilde{A}$ is A-matched to $H \cap B$. Let $\tilde{A} \cap \langle U_0 S \rangle \neq \{0\}$ and choose a non-zero element a of it. We extend $\{a\}$ to a basis $\{a, a_2, \ldots, a_m\}$ for $\tilde{A}$. Then, there exists a basis $\{b, b_2, \ldots, b_m\}$ of $H \cap B$ such that $ab \notin A$ and $a_i b_i \notin A$, where $2 \leq i \leq m$ (according to local matchability). But, we have $ab \in \langle U_0 S \rangle H = \langle U_0 S \rangle \subseteq A$, which contradicts the case $\tilde{A}$ is A-matched to $H \cap B$. So $\tilde{A} \cap U_0 S = \{0\}$. Then, $\dim_K \tilde{A} + \dim_K \langle U_0 S \rangle \leq n$. This yields $\dim_K \langle H \cap U \rangle + \dim_K \langle (U \cup \{1\} S) \rangle \leq n$. This follows $\dim_K \langle H \cap U \rangle + \dim_K \langle S \cup SU \rangle \leq n$. So, by (18) we have $\dim_K U + \dim_K S \leq n$, which is impossible.

Then in both cases, we extract contradictions and so A is matched to B. $\square$

Using Theorems 5.2 and 5.3, we give an alternative proof to the auther's main theorem in [6] which states that the necessary and sufficient condition for a field extension having linear matching property is not having any proper intermediate field with finite degree. However, our proof works just in the case that every algebraic element is assumed to be separable. Note that we use this fact that matchability and local matchability are equivalent in such a field extension. Let $K \subseteq L$ be a field extension whose algebraic elements are separable and has no proper intermediate field with finite degree. If A and B are two n-dimensional K-subspaces of L with $n \geq 1$ and $1 \notin B$, clearly A is locally matched to B and then A is matched to B. This means $K \subseteq L$ has

the linear matching property.

Coversely, assume that $K \subseteq L$ has the proper intermediate field H of finite degree. Set $A = H$, choose $x \in L \setminus H$ and define $B = \langle H \cup \{x\} \setminus K \rangle$. Clearly $\dim_K A = \dim_K B$, $H \cap B \neq \{0\}$ and $aH \subseteq H = A$, for some $a \in A$. (Indeed for any $a \in A$). If A is locally matched to B, then a K-subspace $\tilde{A}$ of A is A-matched to $H \cap B$. If $\{a_1, \ldots, a_m\}$ is a basis for $\tilde{A}$, then there must be a basis $\{b_1, \ldots, b_m\}$ of $H \cap B$ for which $a_i b_i \notin A$, for $1 \leq i \leq m$. But this is impossible because $a_i b_i \in A(H \cap B) \subseteq H\langle H \setminus K \rangle \subseteq H = A$. Then, A is not locally matched to B and so A is not matched to B. Thus $K \subseteq L$ does not have the linear matching property.

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